

EINSTEIN WARPED PRODUCT MANIFOLDS WITH 3– DIMENSIONAL FIBER MANIFOLDS

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ABSTRACT. In this paper, we consider the existence of nonconstant warping functions on a warped product manifold $M = B \times_{f^2} F$, where B is a $q(> 2)$ –dimensional base manifold with a nonconstant scalar curvature $S_B(x)$ and F is a 3–dimensional fiber Einstein manifold and discuss that the resulting warped product manifold is an Einstein manifold, using the existence of the solution of some partial differential equation.

1. Introduction

In [2], A.L. Besse studied a new compact Einstein manifold using the warped product. Then A.L. Besse asked the following: “Does there exist an Einstein warped product manifold with a nonconstant warping function?”

In [9],[10], and [11], the authors proved that there does not exist a compact Einstein warped product space with a nonconstant warping function, if the scalar curvature on M is nonpositive or the base is a compact 2–dimensional manifold. Hence here we assume that the base manifold B is a compact $q(> 2)$ –dimensional manifold with the positive scalar curvature somewhere.

DEFINITION 1.1. Let (B, g_B) and (F, g_F) be two manifolds. Let g_B be a metric tensor of B and g_F be a metric tensor of F . We denote by π and σ the projections of $B \times F$ onto B and F , respectively. For a positive smooth function f on B the warped product manifold $M = B \times_{f^2} F$ is the product manifold $M = B \times F$ furnished with the metric tensor g

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defined by $g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F)$. We denote by π^* and σ^* the pullback π and σ , respectively. Here B is called the base of M and F the fiber([1,3,4,5,12]).

Now we recall the formula for the Ricci curvature tensor Ric of the warped product manifold $M = B \times_{f^2} F$. We write Ric^B for the pullback by π of the Ricci curvature of B and similarly for Ric^F .

PROPOSITION 1.2. *On a warped product manifold $M = B \times_{f^2} F$ with $p = \dim F > 1$, let X, Y be horizontal and V, W vertical. Then*

- (i) $Ric(X, Y) = Ric^B(X, Y) - \frac{p}{f} H^f(X, Y)$,
- (ii) $Ric(X, V) = 0$,
- (iii) $Ric(V, W) = Ric^F(V, W) - g(V, W) \left(\frac{\Delta f}{f} + (p - 1) \frac{g(df, df)}{f^2} \right)$,

where H^f and Δf denote by the Hessian of f and the Laplacian of f for g_B .

Proof. See Proposition 9.106 in ([2, p.266]). □

COROLLARY 1.3. *Let F be a 3 - dimensional manifold. The warped product $M = B \times_{f^2} F$ is an Einstein manifold (with $Ric = \lambda g$) if and only if g_F, g_B and f satisfy*

- (i) (F, g_F) is Einstein (with $Ric_F = \lambda_0 g_F$),
- (ii) $\frac{\Delta f}{f} - 2 \frac{\|df\|^2}{f^2} + \frac{\lambda_0}{f^2} = \lambda$,
- (iii) $Ric_B - \frac{3}{f} H^f = \lambda g_B$.

Proof. See Corollary 9.107 in [2, p.267] □

Obviously, (i) gives a condition on (F, g_F) alone, whereas (ii) and (iii) are two differential equations for f on (B, g_B) .

In this paper, we consider the following question:

Question A : If the base manifold B is a compact $q(> 2)$ - dimensional manifold and the fiber manifold F is a 3 - dimensional Einstein manifold with $Ric_F = \lambda_0 g_F$, then do there exist a constant λ and a nonconstant warping function f such that the resulting warped product manifold $M = B \times_{f^2} F$ is an Einstein manifold with $Ric = \lambda g$?

In [6], the author proved that if B is a compact $q(> 2)$ - dimensional manifold with a nonconstant scalar curvature and F is a $p(> 3)$ - dimensional Einstein manifold, then there exist a constant λ and a

nonconstant warping function f such that the resulting warped product manifold $M = B \times_{f^2} F$ is an Einstein manifold with $Ric = \lambda g$.

In this paper, the similar results are proved in case of 3 - dimensional fiber manifolds, using a partial differential equation.

REMARK 1.4. We denote by $\dim B = q (> 2)$ and $\dim F = 3$. Then, using Corollary 1.3 (ii) and (iii), we may replace the unique equation

$$(1.1) \quad Ric_B - \frac{3}{f} H^f = \frac{1}{2} \left[S_B + 6 \frac{\Delta f}{f} - 6 \frac{\|df\|^2}{f^2} + 3 \frac{\lambda_0}{f^2} - (1+q)\lambda \right] g_B,$$

where S_B is a scalar curvature of B (See also (9.108) in [2, p.267]).

In order to solve Question A, we study equation (1.1) on M with $\dim B = q (> 2)$ and $\dim F = 3$. Recalling that $Ric_B - \frac{3}{f} H^f = \lambda g_B$ and that S_B is a scalar curvature on B , equation (1.1) implies that we have equation

$$(1.2) \quad 0 = \Delta f - \frac{\|df\|^2}{f} + \frac{\lambda_0}{2f} + \frac{S_B - (3+q)\lambda}{6} f.$$

From now on, we study the nonconstant solution of equation (1.2). If $S_B = C$ is constant, then the solution f is maybe also constant. In case that S_B is not a constant, then the solution also is not a constant. So we assume that S_B is not a constant.

Using the change of variable $f = e^{-u}$, equation (1.2) is changed into

$$(1.3) \quad 0 = \Delta u - \frac{S_B(x) - (3+q)\lambda}{6} - \frac{\lambda_0}{2} e^{2u}.$$

We put $\frac{\lambda_0}{2} = C_{\lambda_0}$ and $\frac{S_B(x) - (3+q)\lambda}{6} = h_\lambda(x)$ (a function depending on $S_B(x)$ and λ). Then equation (1.3) is changed into

$$(1.4) \quad 0 = \Delta u - h_\lambda(x) - C_{\lambda_0} e^{2u},$$

where λ is a constant.

In order to solve equation (1.4), we consider the following functional J_λ for a fixed constant λ , i.e.,

$$J_\lambda(u) = \frac{\int_B |\nabla u|^2 dB - 2 \int_B h_\lambda(x) u dB}{\int_B e^{2u} dB}$$

2. Main results

Let B be a compact connected manifold, which is not necessarily orientable and possesses a given Riemannian structure g . We denote the volume element of this metric by dB , the gradient by ∇ , and the associated Laplacian by Δ (we use the sign convention which gives $\Delta u = -u_{xx} - u_{yy}$ for the standard metric on \mathbb{R}^2). We let $H_{s,r}(B)$ denote the Sobolev space of functions on B whose derivatives through order s are in $L_r(B)$. The norm on $H_{s,r}(B)$ will be denoted by $\|\cdot\|_{s,r}$. In the special case $s = 0$, $H_{s,r}(B)$ is just $L_r(B)$, and we denote the norm by $\|\cdot\|_r$. We have the following elementary inequality.

LEMMA 2.1. For all $v \in H_{1,2}(B)$, if $v \neq 0$ and $\int_B v dB = 0$, then

$$(2.1) \quad \int_B e^v dB > \text{vol}(B),$$

where $\text{vol}(B)$ is the volume of B .

Proof. See [7, Theorem 2.5 and Corollary 1, p.3228]. □

By Lemma 2.1, if we choose a function $v \in H_{1,2}(B)$ and v is not a constant, then $\int_B e^{v-\bar{v}} dB > \text{vol}(B)$, where $\bar{v} = \int_B v dB$. Hence we can consider the functional J_λ on $V_\sigma = \{v \in H_{1,2}(B) \mid v \neq 0, \int_B v dB = 0, \int_B e^{2v} dB = \sigma\}$ for some constant $\sigma (> \text{vol}(B))$,

$$\begin{aligned} J_\lambda(v) &= \frac{\int_B |\nabla v|^2 dB - 2 \int_B h_\lambda(x)v dB}{\int_B e^{2v} dB} \\ &= \frac{1}{\sigma} \left[\int_B |\nabla v|^2 dB - 2 \int_B h_\lambda(x)v dB \right]. \end{aligned}$$

THEOREM 2.2. For a fixed constant λ , let $\{v_i\}$ be a minimizing sequence in V_σ such that $J_\lambda(v_i) \rightarrow C$ for some constant C . If $v_i \rightarrow v_0$ in V_σ and $J_\lambda(v_0) = C$, then equation (1.4) has a solution v_0 for some constant C .

Proof. For a fixed constant λ , let v_0 satisfy

$$J_\lambda(v_0) = \frac{\int_B |\nabla v_0|^2 dB - 2 \int_B h_\lambda(x)v_0 dB}{\int_B e^{2v_0} dB} = C.$$

For all $\psi \in H_{1,2}(B)$,

$$\begin{aligned}
& \frac{dJ_\lambda(v_0 + t\psi)}{dt} \Big|_{t=0} \\
&= \frac{d}{dt} \left[\frac{\int_B (|\nabla v_0 + t\nabla\psi|^2 - 2h_\lambda(x)(v_0 + t\psi)) dB}{\int e^{2(v_0+t\psi)} dB} \right] \Big|_{t=0} \\
&= \frac{1}{(\int_B e^{2(v_0+t\psi)} dB)^2} \left[\int_B 2\nabla v_0 \nabla\psi dB - 2 \int_B h_\lambda(x)\psi dB \right. \\
&\quad \left. + 2t \int_B |\nabla\psi|^2 dB \right] \left\{ \int_B e^{2(v_0+t\psi)} dB \right\} \\
&\quad - \left\{ \int_B |\nabla v_0 + t\nabla\psi|^2 dB - 2 \int_B h_\lambda(x)(v_0 + t\psi) dB \right\} \\
&\quad \times \left\{ 2 \int_B e^{2(v_0+t\psi)} \psi dB \right\} \Big|_{t=0} \\
&= \frac{1}{(\int_B e^{2(v_0)} dB)^2} \left[\left\{ 2 \int_B \nabla v_0 \nabla\psi dB - 2 \int_B h_\lambda(x)\psi dB \right\} \right. \\
&\quad \times \left\{ \int_B e^{2(v_0)} dB \right\} - \left\{ \int_B |\nabla v_0|^2 dB - 2 \int_B h_\lambda(x)v_0 dB \right\} \\
&\quad \left. \times \left\{ 2 \int_B e^{2(v_0)} \psi dB \right\} \right] = 0.
\end{aligned}$$

Therefore

$$\int_B \nabla v_0 \nabla\psi dB - \int_B h_\lambda(x)\psi dB - C \int_B e^{2(v_0)} \psi dB = 0,$$

for all $\psi \in H_{1,2}(B)$. Since Δ is the negative Laplacian, we have

$$(2.2) \quad \Delta v_0 - h_\lambda(x) - C e^{2v_0} = 0,$$

where C is a constant. □

For $v \in H_{1,2}(B)$, let $v^+(x) = \max\{0, v(x)\}$ and $v^-(x) = \min\{0, v(x)\}$. Then we know easily that $2v^+(x) \leq e^{2v^+(x)+2v^-(x)} = e^{2v(x)}$ for each x , hence we have the following key lemma.

LEMMA 2.3. *For a fixed constant λ , if $v \in V_\sigma$, then $|\int_B h_\lambda(x)v dB| \leq N_0\sigma$, where $N_0 = \max_{x \in B} |h_\lambda(x)|$.*

Proof. If $v \in V_\sigma$ and $\int_B v dB = 0$, then $\int_B |v| dB = 2 \int_B v^+ dB$. Hence $\int_B |v| dB \leq \int_B e^{2v} dB$. Thus

$$\left| \int_B h_\lambda(x)v dB \right| \leq N_0\sigma,$$

where $N_0 = \max_{x \in B} |h_\lambda(x)|$. \square

THEOREM 2.4. On $V_\sigma = \{v \in H_{1,2}(B) | v \neq 0, \int_B v dB = 0, \int_B e^{2v} dB = \sigma\}$ for some constant $\sigma (> \text{vol}(B))$, the functional $J_\lambda(v)$ is bounded below for a fixed constant λ .

Proof. Since B is compact, $\max_{x \in B} |h_\lambda(x)| \leq N_0$ for some positive constant N_0 . If $v \in V_\sigma$, then by Lemma 2.3

$$\begin{aligned} J_\lambda(v) &\geq \frac{1}{\sigma} \left[\int_B |\nabla v|^2 dB - 2N_0 \int_M |v| dB \right] \\ &\geq -2N_0. \end{aligned}$$

This means that $J_\lambda(v)$ is bounded below on V_σ . \square

We consider the following functional J_λ for a fixed constant λ on $V_\sigma = \{v \in H_{1,2}(B) | v \neq 0, \int_B v dB = 0, \int_B e^{2v} dB = \sigma\}$ for some constant $\sigma (> \text{vol}(B))$,

$$J_\lambda(v) = \frac{\int_B |\nabla v|^2 dB - 2 \int_B h_\lambda(x) v dB}{\int_B e^{2v} dB} = \frac{1}{\sigma} \left[\int_B |\nabla v|^2 dB - 2 \int_B h_\lambda(x) v dB \right].$$

THEOREM 2.5. Let $C = \inf_{v \in V_\sigma} J_\lambda(v)$ for a fixed constant λ and for some constant $\sigma (> \text{vol}(B))$. If $C_{\lambda_0} = \frac{\lambda_0}{2} = C$, then there exists a nonconstant solution of equation (1.4) for $C_{\lambda_0} = \frac{\lambda_0}{2} = C$.

Proof. Since $h_\lambda(x)$ is smooth on B , Theorem 2.4 implies that J_λ is bounded below on V_σ . Hence there exists a minimizing sequence $\{v_i\}$ in V_σ such that $J_\lambda(v_i) \rightarrow C$. Because V_σ is not empty, there is some $v_1 \in V_\sigma$. Hence there is a $b > 0$ such that $J_\lambda(v_1) < b$ and $J_\lambda(v_n) \leq b$ for all n .

For $v_n \in V_\sigma$,

$$\sigma J_\lambda(v_n) = \int_B |\nabla v_n|^2 dB - 2 \int_B h_\lambda(x) v_n dB \geq \int_B |\nabla v_n|^2 dB - 2N_0\sigma.$$

Hence $\int_B |\nabla v_n|^2 dB \leq (b + 2N_0)\sigma$. It follows that $\|v_n\|_{1,2}^2 \leq \text{constant}$ for all n . Since the unit ball in any Hilbert space is weakly compact ([1,p.74]), there exist a subsequence $\{v_i\}$ of $\{v_n\}$ and a function $v_0 \in H_{1,2}(B)$ such that :

i) $v_i \rightarrow v_0$ strongly in $L_2(B)$

- ii) $v_i \rightarrow v_0$ weakly in $H_{1,2}(B)$
- iii) $v_i \rightarrow v_0$ pointwise almost everywhere.

This implies that $\int_B e^{2v_0} dB = \sigma$, $\int_B v_0 dB = 0$ and $\int_B h_\lambda(x)v_i dB \rightarrow \int_B h_\lambda(x)v_0 dB$. Therefore $v_0 \in V_\sigma$. Hence $J_\lambda(v_0) \geq C$.

To conclude that v_0 minimizes J_λ for all $v \in V_\sigma$, we use the general result that whenever v_n converges to v_0 weakly in a Hilbert space, then $\|\nabla v_0\|_2 \leq \liminf \|\nabla v_n\|_2$. Thus $J_\lambda(v_0) \leq J_\lambda(v_n)$ for all n and $J_\lambda(v_0) \leq C$. Therefore v_0 minimizes J_λ in V_σ . \square

THEOREM 2.6. *Let $V_\sigma = \{v \in H_{1,2}(B) | v \neq 0, \int_B v dB = 0, \int_B e^{2v} dB = \sigma\}$ for some constant $\sigma (> \text{vol}(B))$. For each λ_0 , there exists a constant λ such that*

$$\inf_{v \in V_\sigma} J_\lambda = \frac{\lambda_0}{2},$$

which implies that Question A holds.

Proof. Since B is compact, the scalar curvature $S_B(x)$ is bounded. Hence $h_\lambda(x) \rightarrow -\infty$ as $\lambda \rightarrow \infty$. And $0 < \delta \leq \inf_{v \in V_\sigma} \frac{\int_B |v| dB}{\int_B e^{2v} dB} \leq 1$, where δ is a positive constant (If $\inf_{v \in V_\sigma} \frac{\int_B |v| dB}{\int_B e^{2v} dB} = 0$, then $\lim \int_B |v_n| dB = 0$, which means a contradiction to the fact that $\sigma = \int_B e^{2v_n} dB \rightarrow \text{vol}(B)$.). Therefore $\inf_{v \in V_\sigma} J_\lambda \rightarrow -\infty$ as $\lambda \rightarrow \infty$. Similarly $\inf_{v \in V_\sigma} J_\lambda \rightarrow +\infty$ as $\lambda \rightarrow -\infty$. Since J_λ is linear with respect to λ , for each λ_0 there exists a constant λ such that

$$\inf_{v \in V_\sigma} J_\lambda = \frac{\lambda_0}{2}.$$

Therefore Theorem 2.5 implies that there exists a nonconstant warping function v_0 such that v_0 is a solution of equation (1.4), which implies that the warped product manifold $M = B \times_{f^2} F$ is an Einstein manifold. \square

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